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A class of logarithmically completely monotonic functions and the best bounds in the first Kershaw's double inequality

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Abstract

In the article, the logarithmically complete monotonicity of a class of functions involving Euler's gamma function are proved, a class of the first Kershaw-type double inequalities are established, and the first Kershaw's double inequality and Wendel's inequality are generalized, refined or extended. Moreover, an open problem is posed.

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1. Introduction

It is well known that the classical Euler's gamma function Γ can be defined for $x > 0$ as $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt$. The digamma or psi function ψ is defined as the logarithmic derivative of Γ and $\psi^{(i)}$ for $i \in \mathbb{N}$ are called polygamma functions.

The ratio $\Gamma(s)/\Gamma(r)$ has been researched by many mathematicians in the past more than fifty years. Wendel [30] gave for $0 < b < 1$ and $x > 0$ the following double inequality:

$$\left(\frac{x}{x+b}\right)^{1-b} \leqslant \frac{\Gamma(x+b)}{x^b \Gamma(x)} \leqslant 1. \tag{1}$$

Gautschi showed in [8] for $0 < s < 1$ and $n \in \mathbb{N}$ that

$$n^{1-s} < \frac{\Gamma(n+1)}{\Gamma(n+s)} < \exp[(1-s)\psi(n+1)]. \tag{2}$$

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A strengthened upper bound was given by Erber in [7]:

$$\frac{\Gamma(n+1)}{\Gamma(n+s)} < \frac{4(n+s)(n+1)^{1-s}}{4n+(s+1)^2}, \quad 0<s<1, \quad n \in \mathbb{N}. \tag{3}$$

Kečkić and Vasić gave in [12] the inequalities below:

$$\frac{b^{b-1}}{a^{a-1}} \cdot e^{a-b} < \frac{\Gamma(b)}{\Gamma(a)} < \frac{b^{b-1/2}}{a^{a-1/2}} \cdot e^{a-b}, \quad 0<a<b. \tag{4}$$

The following closer bounds were proved for $0<s<1$ and $x \geqslant 1$ by Kershaw in [13]:

$$\left(x+\frac{s}{2}\right)^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \left[x-\frac{1}{2}+\left(s+\frac{1}{4}\right)^{1/2}\right]^{1-s}, \tag{5}$$

$$\exp[(1-s)\psi(x+s^{1/2})] < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \exp\left[(1-s)\psi\left(x+\frac{s+1}{2}\right)\right]. \tag{6}$$

Let s and t be nonnegative numbers, $\alpha=\min\{s,t\}$, and

$$z_{s,t}(x) = \begin{cases} \left[\dfrac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)} - x, & s \neq t, \\ e^{\psi(x+s)} - x, & s=t \end{cases} \tag{7}$$

in $x \in (-\alpha, \infty)$. In [5,6,27], a monotonicity and convexity of $z_{s,t}(x)$ was obtained: The function $z_{s,t}(x)$ is either convex and decreasing for $|t-s|<1$ or concave and increasing for $|t-s|>1$. From this, the best bounds in the first Kershaw's double inequality (5) were deduced.

For a and b being two constants, as $x \to \infty$, the following asymptotic formula is given in [1, pp. 257, 259]:

$$x^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)} = 1 + \frac{(a-b)(a+b-1)}{2x} + \mathrm{O}\left(\frac{1}{x^2}\right). \tag{8}$$

For recent development and more detailed information on this topic, please refer to, for example, [5,6,16,27] and the references therein.

Recall [18,31] that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and $(-1)^n f^{(n)}(x) \geqslant 0$ for $x \in I$ and $n \geqslant 0$. Recall [23–25] also that a function f is called logarithmically completely monotonic on an interval I if f has derivatives of all orders on I and its logarithm $\ln f$ satisfies $0 \leqslant (-1)^k[\ln f(x)]^{(k)}$ for all $k \in \mathbb{N}$ on I. For our own convenience, the sets of the completely monotonic functions and the logarithmically completely monotonic functions on I are denoted, respectively, by $\mathscr{C}[I]$ and $\mathscr{L}[I]$. In [3,18,23–26], it has been proved that $\mathscr{L}[I] \subset \mathscr{C}[I]$. The well-known Bernstein's Theorem [31, p. 161] states that $f \in \mathscr{C}[(0,\infty)]$ if and only if there exists a bounded and nondecreasing function $\eta(t)$ such that the integral $f(x)=\int_0^\infty e^{-xt}\,\mathrm{d}\eta(t)$ converges for $0<x<\infty$. In [3, Theorem 1.1, 9] it is pointed out that the logarithmically completely monotonic functions on $(0,\infty)$ can be characterized as the infinitely divisible completely monotonic functions studied by Horn in [11, Theorem 4.4]. For more information on the classes $\mathscr{C}[I]$ and $\mathscr{L}[I]$, please refer to [2,3,9,18,20,23–27,29] and the references therein.

For $x>0$ and $a>0$, let

$$h_a(x) = \frac{(x+a)^{1-a}\Gamma(x+a)}{x\Gamma(x)} \quad \text{and} \quad f_a(x) = \frac{\Gamma(x+a)}{x^a\Gamma(x)}, \tag{9}$$

where Γ is the classical Euler's gamma function. In [21,22], among other things, the logarithmically completely monotonic properties of the functions $h_a(x)$ and $f_a(x)$ are obtained:

(1) $\lim_{x\to 0+} h_a(x) = \Gamma(a+1)/a^a$ and $\lim_{x\to\infty} h_a(x)=1$ for any $a>0$;
(2) $h_a(x) \in \mathscr{L}[(0,\infty)]$ if $0<a<1$;
(3) $[h_a(x)]^{-1} \in \mathscr{L}[(0,\infty)]$ if $a>1$;
(4) $\lim_{x\to\infty} f_a(x)=1$ for any $a \in (0,\infty)$;

(5) $f_a(x) \in \mathscr{L}[(0,\infty)]$ and $\lim_{x\to 0+} f_a(x) = \infty$ if $a > 1$;
(6) $[f_a(x)]^{-1} \in \mathscr{L}[(0,\infty)]$ and $\lim_{x\to 0+} f_a(x) = 0$ if $0 < a < 1$.

Observe that the functions $h_a(x)$ and $f_a(x)$ can be rewritten as

$$h_a(x) = (x+a)^{1-a}\frac{\Gamma(x+a)}{\Gamma(x+1)} \quad \text{and} \quad f_a(x) = x^{1-a}\frac{\Gamma(x+a)}{\Gamma(x+1)}. \tag{10}$$

In [4], the function $(\Gamma(x+1)/\Gamma(x+s))(x+s/2)^{s-1}$ for $s \in (0,1)$ is proved to be completely monotonic in $(0,\infty)$.

These hint us to consider the logarithmically complete monotonicity of the function

$$H_{a,b,c}(x) = (x+c)^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)} \tag{11}$$

for $x \in (-\rho,\infty)$, where a, b and c are real numbers and $\rho = \min\{a,b,c\}$.

The first main result of this paper is the following Theorem 1.

Theorem 1. *Let a, b and c be real numbers and $\rho = \min\{a,b,c\}$. Then*

(1) *$H_{a,b,c}(x) \in \mathscr{L}[(-\rho,\infty)]$ if $(a,b,c) \in D_1(a,b,c)$, where*

$$\begin{aligned} D_1(a,b,c) = {} & \{(a,b,c) : a+b \geqslant 1, c \leqslant b < c + \tfrac{1}{2}\} \cup \{(a,b,c) : a > b \geqslant c + \tfrac{1}{2}\} \\ & \cup \{(a,b,c) : 2a+1 \leqslant a+b \leqslant 1, a < c\} \cup \{(a,b,c) : b-1 \leqslant a < b \leqslant c\} \\ & \setminus \{(a,b,c) : a = c+1, b = c\}. \end{aligned} \tag{12}$$

(2) *$[H_{a,b,c}(x)]^{-1} \in \mathscr{L}[(-\rho,\infty)]$ if $(a,b,c) \in D_2(a,b,c)$, where*

$$\begin{aligned} D_2(a,b,c) = {} & \{(a,b,c) : a+b \geqslant 1, c \leqslant a < c + \tfrac{1}{2}\} \cup \{(a,b,c) : b > a \geqslant c + \tfrac{1}{2}\} \cup \{(a,b,c) : b < a \leqslant c\} \\ & \cup \{(a,b,c) : b+1 \leqslant a, c \leqslant a \leqslant c+1\} \cup \{(a,b,c) : b+c+1 \leqslant a+b \leqslant 1\} \\ & \setminus \{(a,b,c) : a = c+1, b = c\} \setminus \{(a,b,c) : b = c+1, a = c\}. \end{aligned} \tag{13}$$

As a direct consequence of the monotonicity of $H_{a,b,c}(x)$ and a generalization and a refinement of the first Kershaw's double inequality (5), the following Theorem 2, the second main result of this paper, is established.

Theorem 2. *Let a, b and c be real numbers, $\rho = \min\{a,b,c\}$ and δ be a constant greater than $-\rho$. If $(a,b,c) \in D_1(a,b,c)$, then inequality*

$$(x+c)^{a-b} < \frac{\Gamma(x+a)}{\Gamma(x+b)} \tag{14}$$

holds in $x \in (-\rho,\infty)$ and inequality

$$\frac{\Gamma(x+a)}{\Gamma(x+b)} \leqslant \frac{\Gamma(\delta+a)}{\Gamma(\delta+b)}\left(\frac{x+c}{\delta+c}\right)^{a-b} \tag{15}$$

sounds in $x \in [\delta,\infty)$. If $(a,b,c) \in D_2(a,b,c)$, then inequalities (14) *and* (15) *are reversed in $(-\rho,\infty)$ and $[\delta,\infty)$, respectively.*

Remark 1. Let us take $a = 1$ and $0 < b < 1$ in inequality (14). Then inequality

$$(x+c)^{1-b} < \frac{\Gamma(x+1)}{\Gamma(x+b)} \tag{16}$$

is valid in $(-\rho, \infty)$ for $(b, c) \in D_1(1, b, c) \cap \{(b, c) : 0 < b < 1\} = \{0 < b < 1, c \leqslant b < 1\} \setminus \{(0, 0)\}$. This implies that, in particular, inequality

$$(x+b)^{1-b} < \frac{\Gamma(x+1)}{\Gamma(x+b)} \tag{17}$$

is valid in $(-b, \infty)$ for $0 < b < 1$.

It is clear that inequality (17) not only refines the lower bound but also extends the range of the argument x of the left-hand side in inequality (5).

Remark 2. Now let us take $a = 1$, $0 < b < 1$ and $\delta = 1$ in inequality (15). Then inequality

$$\frac{\Gamma(x+1)}{\Gamma(x+b)} \leqslant \frac{1}{\Gamma(1+b)} \left(\frac{x+c}{1+c} \right)^{1-b} \tag{18}$$

validates in $[1, \infty)$ for $(b, c) \in D_1(1, b, c) \cap \{(b, c) : 0 < b < 1\} \cap \{(b, c) : -\rho < 1\} = \{0 < b < 1, c \leqslant b < 1\} \cap \{(b, c) : -\rho < 1\} \setminus \{(0, 0)\} = \{(b, c) : 0 < b < 1, -1 < c \leqslant b < 1\}$. In particular, for $0 < b < 1$, inequality

$$\frac{\Gamma(x+1)}{\Gamma(x+b)} \leqslant \frac{1}{\Gamma(1+b)} \left(\frac{x+b}{1+b} \right)^{1-b} \tag{19}$$

makes sense in $x \in [1, \infty)$.

Standard argument reveals that if

$$x \geqslant \frac{(1/2 - \sqrt{b + 1/4})(1+b)\sqrt[1-b]{\Gamma(1+b)} + 1}{(1+b)\sqrt[1-b]{\Gamma(1+b)} - 1} \triangleq \lambda(b) \tag{20}$$

then inequality (19) would be better than the right-hand side of (5). It is easy to obtain that $\lim_{b \to 0+} \lambda(b) = \infty$ and

$$\lim_{b \to 1-} \lambda(b) = \frac{e + e^{\gamma} - \sqrt{5}\, e^{\gamma}}{2e^{\gamma} - e} = 0.6123686 \cdots < 1,$$

where $\gamma = 0.57721566\ldots$ is the Euler–Mascheroni's constant. This means that inequality (19) refines the right-hand side of (5) if b is closer enough to 1 and that the upper bound in (19) is better than the one in (5) if x is larger enough.

Remark 3. The inequality (1) can be rewritten as

$$(x+b)^{1-b} \leqslant \frac{\Gamma(x+1)}{\Gamma(x+b)} \leqslant x^{1-b}. \tag{21}$$

It is easy to see that the range of the argument x in inequality (17) is larger than that in the left-hand side of inequality (21).

Taking $a = 1$, $0 < b < 1$ and $\delta = 0$ in inequality (15) yields

$$\frac{\Gamma(x+1)}{\Gamma(x+b)} \leqslant \frac{1}{\Gamma(b)} \left(\frac{x+c}{c} \right)^{1-b} \tag{22}$$

for $(b, c) \in D_1(1, b, c) \cap \{(b, c) : 0 < b < 1\} \cap \{(b, c) : -\rho < 0\} = \{0 < b < 1, c \leqslant b < 1\} \cap \{(b, c) : -\rho < 0\} \setminus \{(0, 0)\} = \{(b, c) : 0 < b < 1, 0 < c \leqslant b < 1\}$. In particular, inequality

$$\frac{\Gamma(x+1)}{\Gamma(x+b)} \leqslant \frac{1}{\Gamma(b)} \left(\frac{x+b}{b} \right)^{1-b} \tag{23}$$

makes true in $[0, \infty)$ for $0 < b < 1$. When

$$x > \frac{1}{b\sqrt[1-b]{\Gamma(b)} - 1}, \tag{24}$$

the upper bound in (23) is better than that in (21).

Remark 4. Since

$$[H_{a,b,c}(x)]^{1/(a-b)} = \frac{1}{x+c}\left[\frac{\Gamma(x+a)}{\Gamma(x+b)}\right]^{1/(a-b)} = \frac{z_{b,a}(x)+x}{x+c} \tag{25}$$

or

$$z_{b,a}(x) = [H_{a,b,c}(x)]^{1/(a-b)}(x+c) - x, \tag{26}$$

the monotonicity and convexity of $z_{b,a}(x)$ and the logarithmically complete monotonicity of $H_{a,b,c}(x)$ are connected.

Remark 5. Eq. (25) shows that $(1+b)\sqrt[1-b]{\Gamma(1+b)}$ in (20) and $b\sqrt[1-b]{\Gamma(b)}$ in (24) can be rewritten as $[H_{1,b,b}(1)]^{1/(b-1)}$ and $[H_{1,b,b}(0)]^{1/(b-1)}$, respectively. The graphs of these two functions, pictured by Mathematica 5.2, remind us that these two functions are increasing in $b \in (-1, \infty)$ and $b \in (0, \infty)$, respectively.

In [16], using some monotonicity results and inequalities of the generalized weighted mean values with two parameters in [10,15,14,17,28], it was verified, among other things, that the functions $[\Gamma(s)/\Gamma(r)]^{1/(s-r)}$ are increasing in both $r > 0$ and $s > 0$. In [27], it was showed that $1/(z_{s,t}(x)+1) \in \mathscr{C}[(-\alpha, \infty)]$.

Now it is natural to pose the following open problem: let $\delta \geqslant 0$, $\lambda \geqslant 0$, μ be a real constant and $k \in \mathbb{N}$ such that $\mu > \lambda(2\delta)^{2k-1}$. For $x, y \in (-\delta, \infty)$, define

$$\Phi_{\delta,\lambda,\mu,k}(x, y) = \begin{cases} \dfrac{1}{\lambda(x+y)^{2k-1}+\mu}\left[\dfrac{\Gamma(\delta+x)}{\Gamma(\delta+y)}\right]^{1/(x-y)}, & x \neq y, \\ \dfrac{e^{\psi(\delta+y)}}{2\lambda y^{2k-1}+\mu}, & x = y. \end{cases} \tag{27}$$

What about the monotonicity, complete monotonicity, logarithmically complete monotonicity or Schur-convexity of the function $\Phi_{\delta,\lambda,\mu,k}(x, y)$?

2. Lemmas

In order to prove our main results, the following lemmas are necessary.

Lemma 1 (*Abramowitz and Stegun [1]*). *For $x > 0$ and $\omega > 0$,*

$$\frac{1}{x^{\omega}} = \frac{1}{\Gamma(\omega)}\int_0^{\infty} t^{\omega-1}e^{-xt}\,\mathrm{d}t. \tag{28}$$

For $k \in \mathbb{N}$ and $x > 0$,

$$\psi(x) = -\gamma + \int_0^{\infty}\frac{e^{-t}-e^{-xt}}{1-e^{-t}}\,\mathrm{d}t. \tag{29}$$

Lemma 2 (*Qi [19,20]*). *For real numbers α and β with $(\alpha, \beta) \notin \{(0, 1), (1, 0)\}$ and $\alpha \neq \beta$, let*

$$q_{\alpha,\beta}(t) = \begin{cases} \dfrac{e^{-\alpha t}-e^{-\beta t}}{1-e^{-t}}, & t \neq 0, \\ \beta - \alpha, & t = 0. \end{cases} \tag{30}$$

(1) *The function $q_{\alpha,\beta}(t)$ is increasing in $(0, \infty)$ if and only if $(\alpha, \beta) \in D_1(\alpha, \beta)$, where*

$$\begin{aligned} D_1(\alpha, \beta) = {} & \{(\alpha, \beta) : \alpha > \beta \geqslant \tfrac{1}{2}\} \cup \{(\alpha, \beta) : \alpha \geqslant 1-\beta, 0 \leqslant \beta < \tfrac{1}{2}\} \cup \{(\alpha, \beta) : \alpha+1 \leqslant \beta \leqslant 1-\alpha, \alpha < 0\} \\ & \cup \{(\alpha, \beta) : \beta - 1 \leqslant \alpha < \beta \leqslant 0\} \backslash \{(1, 0)\}. \end{aligned} \tag{31}$$

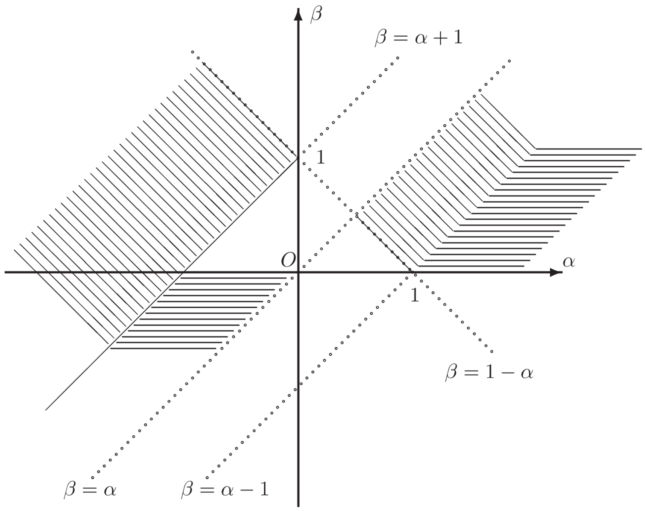

Fig. 1. The (α, β)-domain $D_1(\alpha, \beta)$ in Lemma 2.

(2) *The function* $q_{\alpha,\beta}(t)$ *is decreasing in* $(0, \infty)$ *if and only if* $(\alpha, \beta) \in D_2(\alpha, \beta)$, *where*

$$\begin{aligned} D_2(\alpha, \beta) = {} & \{(\alpha, \beta) : \beta \geqslant 1-\alpha, \tfrac{1}{2} > \alpha \geqslant 0\} \cup \{(\alpha, \beta) : \beta > \alpha \geqslant \tfrac{1}{2}\} \cup \{(\alpha, \beta) : \beta < \alpha \leqslant 0\} \\ & \cup \{(\alpha, \beta) : \beta \leqslant \alpha - 1, 0 \leqslant \alpha \leqslant 1\} \cup \{(\alpha, \beta) : 1 \leqslant \alpha \leqslant 1-\beta\} \backslash \{(1, 0), (0, 1)\}. \end{aligned} \tag{32}$$

Remark 6. The (α, β)-domains $D_1(\alpha, \beta)$ and $D_2(\alpha, \beta)$ defined in Lemma 2, where the function $q_{\alpha,\beta}(t)$ is increasingly or decreasingly monotonic in $(0, \infty)$, can be described, respectively, by Figs. 1 and 2.

Remark 7. In [19,20,27], the monotonicity, logarithmic convexity and 3-log-convexity of the function $q_{\alpha,\beta}(t)$ in either $(-\infty, 0)$, or $(0, \infty)$, or $(-\infty, \infty)$ have been investigated thoroughly.

3. Proofs of theorems

Proof of Theorem 1. By formulas (28) and (29), direct computation yields

$$\begin{aligned} \ln H_{a,b,c}(x) &= (b-a)\ln(x+c) + \ln\Gamma(x+a) - \ln\Gamma(x+b), \\ [\ln H_{a,b,c}(x)]' &= \frac{b-a}{x+c} + \psi(x+a) - \psi(x+b) \\ &= \frac{b-a}{x+c} + \int_0^\infty \frac{e^{-bt}-e^{-at}}{1-e^{-t}} e^{-xt}\,\mathrm{d}t \\ &= -\int_0^\infty \left[\frac{e^{(c-a)t}-e^{(c-b)t}}{1-e^{-t}} + (a-b)\right] e^{-(x+c)t}\,\mathrm{d}t \\ &= -\int_0^\infty [q_{a-c,b-c}(t) + (a-b)] e^{-(x+c)t}\,\mathrm{d}t \end{aligned}$$

and, for $k \in \mathbb{N}$,

$$(-1)^k[\ln H_{a,b,c}(x)]^{(k)} = \int_0^\infty [q_{a-c,b-c}(t) + (a-b)] t^{k-1} e^{-(x+c)t}\,\mathrm{d}t,$$

where $q_{\alpha,\beta}(t)$ is defined by (30) in Lemma 2.

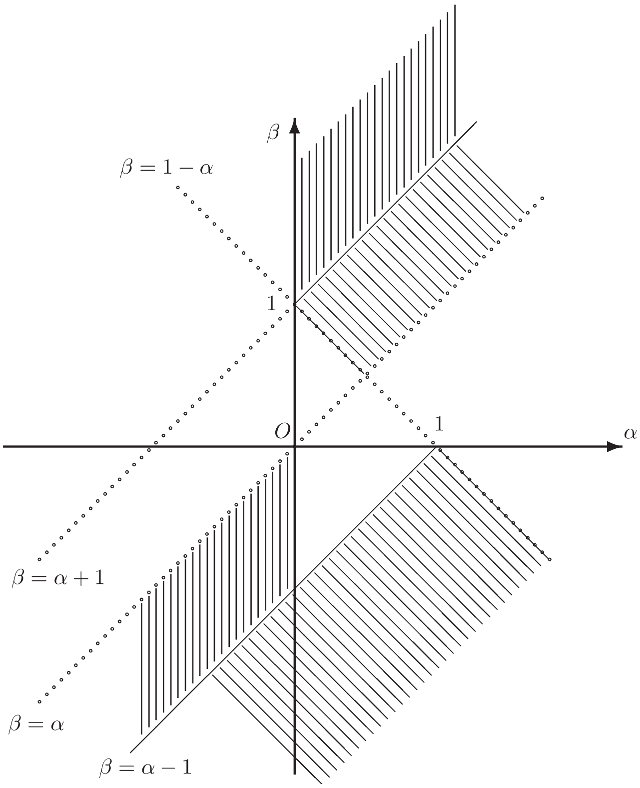

Fig. 2. The (α, β)-domain $D_2(\alpha, \beta)$ in Lemma 2.

From $q_{\alpha,\beta}(0) = \beta - \alpha$ and $q_{a-c,b-c}(0) = b - a$, it is deduced that if $q_{a-c,b-c}(t)$ is increasing (or decreasing, respectively) in $(0, \infty)$ then $q_{a-c,b-c}(t) + (a - b) \gtreqless 0$ in $t \in (0, \infty)$ and $(-1)^k[\ln H_{a,b,c}(x)]^{(k)} \gtreqless 0$ in $x \in (-\rho, \infty)$ for $k \in \mathbb{N}$. Combining this with Lemma 2 reveals that $H_{a,b,c}(x) \in \mathscr{L}[(-\rho, \infty)]$ if $(a - c, b - c) \in D_1(a - c, b - c)$ and $[H_{a,b,c}(x)]^{-1} \in \mathscr{L}[(-\rho, \infty)]$ if $(a - c, b - c) \in D_2(a - c, b - c)$. The proof of Theorem 1 is complete. $\square$

Proof of Theorem 2. By formula (8), it follows that

$$
\begin{aligned}
H_{a,b,c}(x) &= \left(1 + \frac{c}{x}\right)^{b-a} \left[x^{b-a} \frac{\Gamma(x + a)}{\Gamma(x + b)}\right] \\
&= \left(1 + \frac{c}{x}\right)^{b-a} \left[1 + \frac{(a - b)(a + b - 1)}{2x} + \mathrm{O}\left(\frac{1}{x^2}\right)\right] \\
&\to 1
\end{aligned}
$$

as $x \to \infty$ for all real numbers a, b and c.

If $(a, b, c) \in D_1(a, b, c)$, the function $H_{a,b,c}(x)$ is decreasing in $(-\rho, \infty)$ and $H_{a,b,c}(x) > \lim_{x\to\infty} = 1$ which can be rearranged as inequality (14). Further, if δ is a constant greater than $-\rho$, then

$$
H_{a,b,c}(x) \leqslant H_{a,b,c}(\delta) = (\delta + c)^{b-a} \frac{\Gamma(\delta + a)}{\Gamma(\delta + b)}
$$

in $[\delta, \infty)$, which can be rewritten as (15) for $x \in [\delta, \infty)$.

If $(a, b, c) \in D_2(a, b, c)$ and δ is also a constant greater than $-\rho$, then the function $H_{a,b,c}(x)$ is increasing in $(-\rho, \infty)$, inequalities (14) and (15) are reversed, respectively. The proof of Theorem 2 is complete. $\square$

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